

Edge Union of Networks on the Same Vertex Set.

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Abstract. Random networks generators like Erdős-Rényi, Watts-Strogatz and Barabási-Albert models are used as models to study real-world networks. Let $G^1(V, E_1)$ and $G^2(V, E_2)$ be two such networks on the same vertex set V . This paper studies the degree distribution and cluster coefficient of the resultant networks, $G(V, E_1 \cup E_2)$.

1. Introduction

Random networks are used as models to study real-world networks like the Internet and social networks. The major families of random networks generators are Erdős-Rényi, Watts-Strogatz and Barabási-Albert models. These models generate a network with only one type of edges and hence may be thought of as relating to agents interacting under a homogeneous relation [1, 2, 3].

However recently research has started to address the limitations of homogeneous network. In the real world, agents may be related through more than one form of relations. E.g. think of social interactions between people, one type of relations may consist of family bonds another of shared interests, such as chess a third could be professional relations and so forth [4, 5, 6, 7]. Networks in which edge represent different relationships are known as multi-dimensional network[‡]. An abstraction is to decompose the network into “layers” of relations (dimensions). Each layer has the same set of vertices, and at every layer the vertices are connected differently.

There are various proposals to solve classic problems like link prediction and communities detection in multi-dimensional network. However none was on the structural property of the whole network. Given the statistical properties of two layers, we address in this paper the statistical properties of the union of the different types of edges. Combining edges of different types is known as the *edge union of networks on the same vertex set*.

For example, a union of Watts-Strogatz and Barabási-Albert can represent union of the transport network of US roadmap (Grid-like) and the US Air traffic (Scale-Free) respectively. The union models the coverage of a courier company within the USA.

Another motivation of this paper can be viewed from a different perspective. Think of the network of email exchanges in a social network. Assume that the preferential attachment of Barabási-Albert model is true in this context, where a new member tends to email the more popular members. However being “popular” at a personal level is different from being “popular” in the work environment. If the database of email exchanges network does not distinguish the two contexts, the email exchanges network is essentially the union of two Barabási-Albert networks.

The present paper is a study of some of the statistical properties of a random network that is formed by the union of two random networks. The different families of random network are distinguished by their clustering coefficients and the degree distribution of the vertices and we want to determine these specific measures for the union of different combinations of networks. We first consider the degree distributions and then the clustering coefficients.

[‡] All the cited papers use different names for the same model. We use the naming convention of the earliest paper, ergo multi-dimensional network

2. Network Family

This section summarizes the construction of Erdős-Rényi, Watts-Strogatz and Barabási-Albert models. Recall that these random network generators were designed to construct simple graphs. We are going to combine networks with different edge statistics but with the same number of vertices, accordingly we suppress in the notation below the number of vertices in the labeling of the specific network types. For instance, the usual notation for Gilbert Graph is $G_{n,p}$, where n is the size of the vertex set. We simplify the notation to G_p as the size of the vertex set is implied to be n for the networks in the rest of the paper. Lastly, we define V and E be vertex set and edge set respectively.

2.1. Erdős-Rényi

A realisation of an Erdős-Rényi network [8] is selected with equal probability from the set of all possible graphs with $n = |V|$ vertices and $|E|$ edges. However to generate a huge random Erdős-Rényi is difficult. To circumvent this problem one may instead let the number of edges fluctuate slightly and consider a Gilbert graph [9] G_p in which every vertex pair is connected with probability p and p is chosen such that $|V|p \approx |E|$. The slight difference is that Erdős-Rényi has precisely $|E|$ edges while Gilbert graph has approximately $|E|$ edges with high probability.

2.2. Watts-Strogatz

A Watts-Strogatz network [10], $W_{w,q}$ is w (mean degree§) and q (probability of rewiring). The construction begins with a regular ring lattice where each vertex connects to $w/2$ neighbors on each side.

For each vertex $v_i \in V$ and $i < a$, each edge leaving v_i is rewired with probability q . The rewiring replaces $\{v_i, v_a\}$ with $\{v_i, v_b\}$ where v_b is chosen uniformly in V and the resultant network remains a simple network.

The key property of the Watts-Strogatz network is that it varies between a regular ring lattice ($q = 0$) and Erdős-Rényi network ($q = 1$). As $q \rightarrow 1$, Watts-Strogatz can be expressed as a Erdős-Rényi graph, i.e. $W_{w,q} \rightarrow G_{w/(n-1)}$.

2.3. Barabási-Albert

Barabási-Albert network [11] is denoted by B_m where m is number of new edges at each iteration. The network begins with some arbitrary small number of vertices connected randomly.

At each iteration, the network grows by one vertex with m new edges. There is a preferential bias that higher degree vertices are connected to the new edges. Define

§ We replace the commonly used variable k with w to avoid confusion with the variable k commonly used in degree distribution function

$\deg(v_i)$ as the degree of vertex v_i . The probability that the new vertex is connected to vertex v_i is given by:

$$p_i = \frac{\deg(v_i)}{\sum_j \deg(v_j)}$$

3. Definitions and Preliminaries

3.1. Definitions

For the rest of the paper, we will refer to the resultant network as the *composite network*. Hence *network* just refers to either an Erdős-Rényi, Watts-Strogatz or Barabási-Albert network. Furthermore, we will make use of the following abbreviations: \mathcal{C} for the composite network and ER for the Erdős-Rényi, WS for the Watts-Strogatz and BA for the Barabási-Albert models. And below we investigate all 6 pairwise edge unions of these three networks, i.e. $\{ER, ER\}$, $\{ER, WS\}$, $\{ER, BA\}$, $\{WS, WS\}$, $\{WS, BA\}$ and $\{BA, BA\}$.

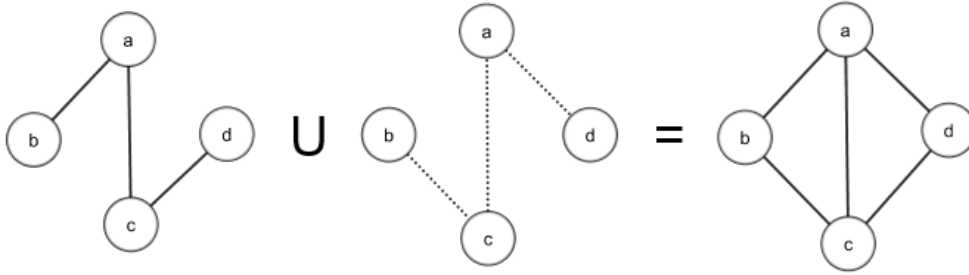


Figure 1: Edge union of two networks on the same vertex set. The solid edges and the dotted edges are from the edge set of different networks. The composite network is the rightmost network. Edge $\{a, c\}$ is a common edge between the two networks.

Definition 3.1. Let $G^1(V, E_1)$ and $G^2(V, E_2)$ be two networks on the same vertex set V . The **edge union of the networks** gives the composite network, $\mathcal{C}(V, E_1 \cup E_2)$.

Definition 3.2. Let $G^1(V, E_1)$ and $G^2(V, E_2)$ be two networks on the same vertex set V . An edge $e \in E_1 \cup E_2$ in the composite network is called a **common edge** if $e \in E_1 \cap E_2$. Hence the set of common edges is $E_1 \cap E_2$.

The rightmost network in figure 1 is an example of a composite network. The solid and dotted edges represents E_1 and E_2 respectively. The composite network does not distinguish the dotted and solid edges from both networks.

Definition 3.3. Consider a population of n urns. Choose a urns and place a red ball in each. Next choose b urns and place a blue ball in each. The probability that exactly i urns contain one red and one blue ball is given the hypergeometric function

$$\mathcal{H}(i; n, a, b) = \frac{\binom{a}{i} \binom{n-a}{b-i}}{\binom{n}{b}}$$

3.2. Preliminaries

We now list a few lemmas and concepts that will become useful during the later analysis.

Lemma 3.1. *Let E_1 and E_2 be the sets of edges of two networks with n vertices. The probability that there are ϵ common edges is:*

$$P(|E_1 \cap E_2| = \epsilon) = \mathcal{H}(\epsilon; \binom{n}{2}, |E_1|, |E_2|) \quad (1)$$

Proof. Choose $|E_2|$ edges out of a total possible $\binom{n}{2}$ edges and color them blue. Next choose $|E_1|$ edges from the set E_1 , the probability of getting ϵ blue edges is defined by the hypergeometric function. \square

We note that equation 1 is the probability that there are ϵ common edges between the two networks. In addition, the expected number of common edges is:

Corollary 3.1. *Let E_1 and E_2 be the set of edges of two networks with n vertices each. The expected number of common edges is the expectancy of the hypergeometric function in equation 1:*

$$\mathbb{E}[|E_1 \cap E_2|] = \mathbb{E}[\mathcal{H}] = \frac{|E_1| \cdot |E_2|}{\binom{n}{2}} \quad (2)$$

In fact we can generalize the above to compute the number of common cliques in the union. It is useful to count the number of common triangles (3-clique) as it affects the accuracy of the cluster coefficient in the union to two networks.

Lemma 3.2. *Let K_1 and K_2 be the sets of k -cliques of two networks with n vertices. The probability that there are ϵ common k -cliques is:*

$$P(|K_1 \cap K_2| = \epsilon) = \mathcal{H}(\epsilon; \binom{n}{k}, |K_1|, |K_2|) \quad (3)$$

Proof. Choose $|K_2|$ edges out of a total possible $\binom{n}{k}$ edges and color them blue. Next choose $|K_1|$ edges from the set E_1 , the probability of getting ϵ blue edges is defined by the hypergeometric function. \square

Corollary 3.2. *Let K_1 and K_2 be the sets of k -cliques of two networks with n vertices each. The expected number of common k -cliques is the expectancy of the hypergeometric function in equation 3:*

$$\mathbb{E}[|K_1 \cap K_2|] = \mathbb{E}[\mathcal{H}] = \frac{|K_1| \cdot |K_2|}{\binom{n}{k}} \quad (4)$$

To compute the degree distribution of the composite network accurately, we have to minimize any double counting of the common edges. For example in figure 1, vertex a of the composite network has degree 3. It is the sum of the degree of the networks (degree 2 from each network) **minus** the number of common edges ($= 1$) at vertex a .

Lemma 3.3. *Let v_{i,G^1} and v_{i,G^2} be the vertices of networks G^1 and G^2 each with n vertices respectively. Given the degree of the vertices, $\deg(v_{i,G^1}) = d_1$ and $\deg(v_{i,G^2}) = d_2$. The probability that there are ϵ common edges between v_{i,G^1} and v_{i,G^2} is:*

$$P_{ce}(\epsilon|d_1, d_2) = \mathcal{H}(\epsilon; n-1, d_1, d_2) \quad (5)$$

Proof. There are only $n-1$ vertices left for v_{i,G^1} and v_{i,G^2} to connect to. Using the same argument as in lemma 3.1, there are d_1 blue edges and d_2 red edges from v_{i,G^1} and v_{i,G^2} respectively. \square

Corollary 3.3. *Let v_{i,G^1} and v_{i,G^2} be the vertices of networks G^1 and G^2 with n vertices respectively. Given the degree of the vertices, $\deg(v_{i,G^1}) = d_1$ and $\deg(v_{i,G^2}) = d_2$. The expected number of common edges between v_{i,G^1} and v_{i,G^2} is:*

$$\mathbb{E}[\mathcal{H}] = \frac{|d_1| \cdot |d_2|}{n-1} \quad (6)$$

For computational ease in this paper we only consider the case where both networks have approximately the same edge set size. Unequal size edge sets introduce additional complexity to the study, since the larger edge set will dominate the resultant network.

4. Degree Distribution of Edge Union of Networks

4.1. Erdős-Rényi with Erdős-Rényi

Let G_p^1 and $G_{p'}^2$ be two independent Gilbert Graphs. The probability that an edge does not exist in both G_p^1 and $G_{p'}^2$ is $(1-p)(1-p')$, accordingly the probability that the composite network will contain an edge is given by $1 - (1-p)(1-p')$ and we have

$$G_p^1 \cup G_{p'}^2 = G_{1-(1-p)(1-p')} \quad (7)$$

4.2. Erdős-Rényi with Watts-Strogatz

Let G_p and $W_{w,q}$ be an ER network and WS network respectively. As $q \rightarrow 1$, $W_{w,q}$ can be approximated as an ER so in this limit the union will be scribed by Eq. (7).

For $q \rightarrow 0$ the degree distribution of the WS network approaches $D_{WS}(k) = \delta(k-w)$ the union is between a lattice-like network and a random, Poisson, network. Neglecting common edges, a vertex in the composite network will have degree k when the corresponding vertex in ER has degree $k-w$. Hence an approximation to the degree distribution of the composite network is:

$$P^C(k) \sim \frac{(np)^{(k-w)} e^{-np}}{(k-w)!} \quad (8)$$

We can improve the estimate of the degree distribution by explicitly discounting the common edges in the union, to do this will become of increasing importance as q is increased from 0. Let the degree distribution of ER and WS be $P^G(k)$ and $P^W(k)$ respectively. To account for the common edges for $P^C(k)$, we have to consider:

- (i) Probability that a vertex in WS, v_{ws} has degree j ;
- (ii) Probability that a vertex in ER, v_{er} has degree $k + \epsilon - j$;
- (iii) Probability that there are ϵ common edges between v_{er} and v_{ws} .

From lemma 3.3, the probability that there are ϵ common edges between v_{er} and v_{ws} is $P_{ce}(\epsilon|j, k + \epsilon - j)$. Thus the improvement on equation 8 is:

$$P^C(k) \sim \sum_{\epsilon}^k \sum_j^k P_{ce}(\epsilon|j, k + \epsilon - j) P^W(j) P^G(k + \epsilon - j) \quad (9)$$

Since WS is lattice-like, most of the WS vertices have a degree close to w . By assuming $P^W(k) = \delta(k - w)$ we can simplify equation 9 to:

$$P^C(k) \sim \sum_{\epsilon}^w P_{ce}(\epsilon|w, k + \epsilon - w) P^G(k + \epsilon - w) \quad (10)$$

4.3. Erdős-Rényi with Barabási-Albert

It was found that the degree distribution of the BA model often doesn't represent natural occurring networks. This has lead to various generalized BA models [12, 13, 14] in which preferential and random uniform attachment are combined. This construction is similar to the union of ER with BA where ER adds some uniform attachment to BA.

To compute the degree distribution, we can use the method from section 4.2. We have:

$$P^C(k) \sim \sum_{\epsilon}^k \sum_j^k P_{ce}(\epsilon|j, k + \epsilon - j) P^G(j) P^B(k + \epsilon - j) \quad (11)$$

where the approximate degree distribution of BA is $P^B(k) \sim 2m^2 k^{-3}$.

However the hypergeometric function, $P_{ce}(\epsilon|j, k + \epsilon - j)$ is computationally expensive. It is difficult to calculate the sums in Eq. (11) for large values of k and thereby assess how the additional uniform attachment influence the power law form of the BA network. To extract the asymptotic behaviour we now make use of a Fokker-Planck approach. The idea is to begin with the BA network and then in each time step add a new uniformly drawn random edge to the BA network. The new edges are taken from the set of ER edges no-common with the BA set. Since there are nm edges in BA, from corollary 3.1 it follows that there are nmp common edges or $\binom{n}{2}p - nmp$ non-common edges.

Let $u(k, t)$ be the number of vertices of degree k at time step t . At each time step, a new edge will change the degree of 2 vertices. Specifically the number of degree k vertices increases by one if the new edge attach to a degree $k - 1$ vertex. This has probability $u(k - 1, t)/n$. Similarly the number of k vertices will decrease by one if the new edge attach to a degree k vertex. This leads to the following equaiton,

$$u(k, t + 1) = u(k, t) + u(k - 1, t)/n - u(k, t)/n \quad (12)$$

By replacing t and k by continuous variables we obtain a partial differential equation, which will be a good approximation for large values of k .

$$\frac{\partial u}{\partial t} + \frac{1}{n} \frac{\partial u}{\partial k} = 0. \quad (13)$$

Using the initial condition $u(k, 0) = P^B(k) \cdot n \sim 2nm^2k^{-3}$ ($P^B(k) = 0$ for $k < m$), we can solve for $u(k, t)$ at $t = 2 \cdot \binom{n}{2}p - nmp \approx np(n - m)$ (twice the number of non-common edges because there are two vertices that change at each time step) and find,

$$P^C(k) = u(k, t) = \frac{2nm^2}{(k + 2n(p - p^2))^3}. \quad (14)$$

In Fig. 2 we compare this asymptotic expression to the functional form obtained by iterating Eq. 12) and to simulation results. The continuum approximation of Eq. (13) doesn't really have a region of validity since the finite number of nodes limits size of the the degree and the asymptotic limit cannot be reach. However, the iterated solution of the Fokker-Planck equaiton (12) matches the simulations well.

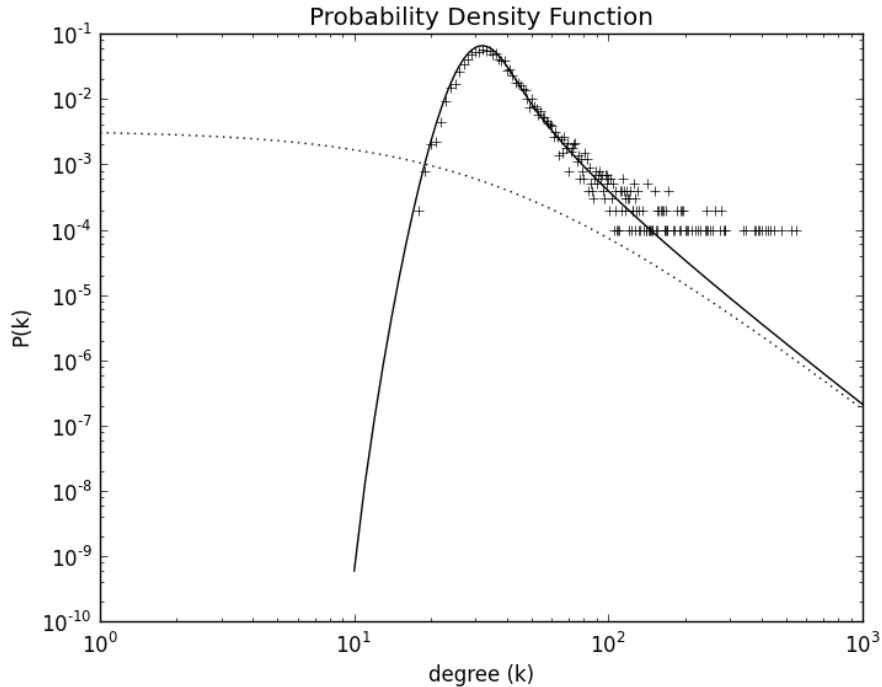


Figure 2: Parameters: $n = 10000$, $m = 10$ and $p = 2m/(n - 1)$ (chosen such that both networks have equal number of edges). The solid line plots the iterative method, Eq. (12). The dotted line plots the closed form Eq. (14). The crosses are actual distribution from simulation.

4.4. Watts-Strogatz with Watts-Strogatz

Let $W_{w,q}^1$ and $W_{w',q'}^2$ be two WS networks. If $q \approx q' \rightarrow 1$, the two WS can be expressed as two ER (section 4.1). In contrast the limit $q \rightarrow 1$ and $q' \rightarrow 0$ is identical to the the

union of ER and WS considered in section 4.2. We do not discuss the case for general values of q and q' .

4.5. Watts-Strogatz with Barabási-Albert

Let $W_{w,q}$ and B_m be WS and BA graphs respectively. There are $wn/2$ edges in WS and nm edges in BA. It is most interesting to consider the case when the graphs have approximately equal number of edges, i.e. $w \approx 2m$. We ignore the case $q \rightarrow 1$, as it is similar to Sec. 4.3.

For $q \rightarrow 0$, the degree distribution of \mathcal{C} is straight forward. Since the probability of rewiring is low, most of the lattice edges in WS remain unchanged. Hence most of the vertices in WS has degree w , and in turn increasing the degree of the vertices of BA by w . This gives the approximation for:

$$P^{\mathcal{C}}(k) \sim 2m^2(k - w)^{-3} \quad (15)$$

Similar to equation 9, the degree distribution can be refined by considering the common edges between WS and BA. However the expression is open form and provide little insights. From simulations ($q \rightarrow 0$), equation 15 is sufficient to approximate the distribution (Figure 3).

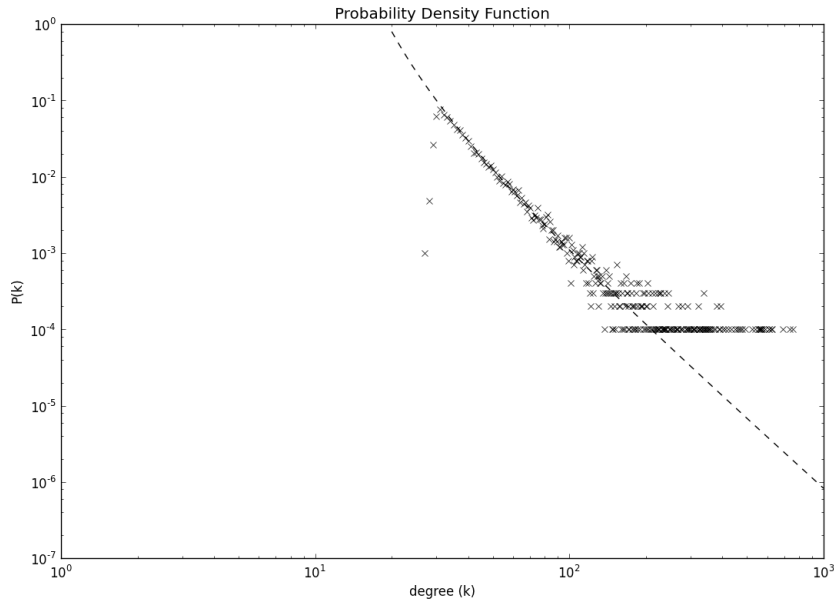


Figure 3: Parameters: $n = 100000$, $w = 10$, $m = 5$ and $q = 0.1$. The crosses are actual degree distribution from simulations. The dashed line plots equation 15.

4.6. Barabási-Albert with Barabási-Albert

The key feature of BA is that the degree distribution is scale-free and follows the power-law distribution. Hence we want to know if the union of two BA retain the features.

Similar to equation 9, the probability density function is the combined probability of:

$$P^C(k) \sim \sum_{\epsilon} \sum_j^k P_{ce}(\epsilon|j, k + \epsilon - j) P^B(j) P^B(k + \epsilon - j) \quad (16)$$

Unfortunately, we are not able to simplify equation 16 to understand the asymptotic behavior of it. However from simulations (figure 4), it appears to behave like a power-law distribution for large k .

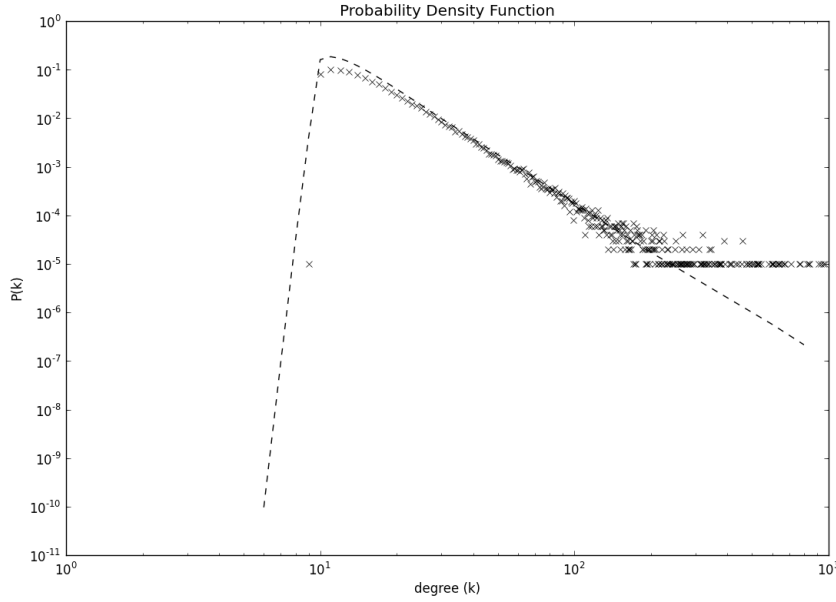


Figure 4: The dashed line plots equation 16 for $n = 100000$ and $m = 5$. Degree distribution of composite network from simulations are plotted with crosses

Unlike section 4.3, we are not able to apply the same method in this combination. Previously we begin with a BA, then we put random edges onto it. The reason why it works is because the edges in ER are independent of each other. In this case, the edges are not independent, due to preferential attachment.

To illustrate this better, let the first BA have red edges, and the second BA to have blue edges. Begin with the BA with red edges, and assume we can put the first few blue edges randomly like ER. However the subsequent blue edges have a preferential bias for **only the blue** edges. Since $u(k,t)$ in section 4.3 does not differentiate the colors on the edges, the method does not apply.

Figure 8 shows the degree rank plot of different combinations of the network union. It is clear that the distribution of ER/BA is slightly different from BA/BA. The union of BA/BA has more high degree vertices than those of BA/ER.

5. Clustering Coefficients of Edge Union of Networks

Let us now consider the cluster coefficient of the composite network. While we are not always able to analytically determine the cluster coefficient, we are able to derive lower

and upper bounds. Lastly we assumed that there are few common edges/triangles in the union, which can be verified with corollary 3.1 and corollary 3.2. This simplification is reasonable for sparse networks and it gives us nice relationships similar to Eq. (17).

5.1. Watts-Strogatz with Watts-Strogatz

The cluster coefficient of such union can be computed analytically in the limit $p \approx q$ and $p, q \rightarrow 0$. Define $T(v_i)$ as the number of subgraphs with 3 edges and 3 vertices (triangles) with vertex v_i . Recall the cluster coefficient equation for a network:

$$\begin{aligned} \text{Cluster Coefficient} &= \frac{1}{n} \sum_{i=0}^n \text{Cluster Coefficient of } v_i \\ &= \frac{1}{n} \sum_{i=0}^n \frac{T(v_i)}{\binom{\deg(v_i)}{2}} \end{aligned}$$

Let v_{i,w^1} and v_{i,w^2} be the vertices of W^1 and W^2 respectively, which are mapped to the i^{th} vertex ($v_{i,c}$) of \mathcal{C} . Since $p \approx q$ and $p, q \rightarrow 0$, almost every vertex of W^1 and W^2 are similar in structure. i.e. $T(v_{i,w^1}) \approx T(v_{i,w^2})$ and $\deg(v_{i,w^1}) \approx \deg(v_{i,w^2})$. Hence even with random pairing, every vertex of the composite network \mathcal{C} will be similar. This gives the analytical computation for small w :

$$\begin{aligned} \text{Cluster Coefficient of } \mathcal{C} &= \frac{1}{n} \sum_{i=0}^n \text{Cluster Coefficient of vertex } v_{i,c} \\ &= \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,c})}{\binom{\deg(v_{i,c})}{2}} \end{aligned}$$

Because w is small ($w \ll n$), we assume there are few common edges. Hence $T(v_{i,c}) = T(v_{i,w^1}) + T(v_{i,w^2}) \approx 2T(v_{i,w^1})$ and $\deg(v_{i,c}) \approx 2\deg(v_{i,w^1})$. Then:

$$\begin{aligned} \text{Cluster Coefficient of } \mathcal{C} &\approx \frac{1}{n} \sum_{i=0}^n \left(\frac{2T(v_{i,w^1})}{\binom{2\deg(v_{i,w^1})}{2}} \right) \\ &\approx \frac{1}{n} \sum_{i=0}^n \frac{1}{2} \left(\frac{T(v_{i,w^1})}{\binom{\deg(v_{i,w^1})}{2}} \right) \\ &\approx \frac{1}{2} \text{ Cluster Coefficient of } W^1 \end{aligned} \tag{17}$$

Figure 5 compares the results of equation 17 with actual simulation of the edge union of two WS. The dots plot the expected cluster coefficient from equation 17 by halving the cluster coefficient of one of the WS. The crosses plot the actual cluster coefficient from simulations. The simulations are detailed in the appendix.

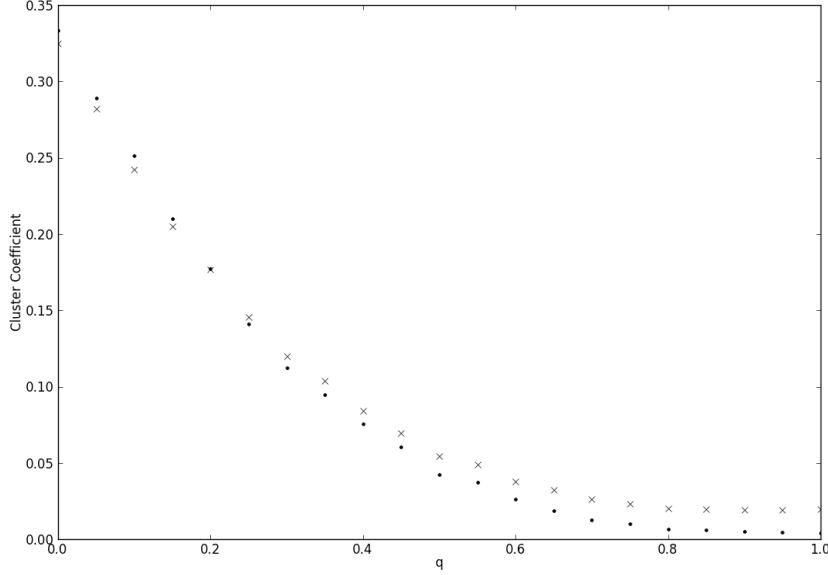


Figure 5: Edge Union of two WS networks for $n = 1000$, $w' = w = 20$ and $q' = q$. The dots are from Eq. (17) and the crosses are from the simulations.

5.2. Watts-Strogatz with Barabási-Albert

The cluster coefficient of the composite is difficult to approximate analytically as there is no analytical result to approximate the number of triangles for BA. Instead, we derive a lower bound and an upper bound for the composite:

$$\begin{aligned} \text{Cluster Coefficient of } \mathcal{C} &= \frac{1}{n} \sum_{i=0}^n \text{Cluster Coefficient of vertex } v_{i,c} \\ &= \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,c})}{\binom{\deg(v_{i,c})}{2}} \end{aligned}$$

Assume $T(v_{i,c}) \approx T(v_{i,ws}) + T(v_{i,ba})$.

$$\begin{aligned} \text{Cluster Coefficient of } \mathcal{C} &\approx \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ws}) + T(v_{i,ba})}{\binom{\deg(v_{i,c})}{2}} \\ &\geq \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ws})}{\binom{\deg(v_{i,c})}{2}} \end{aligned}$$

Since we are considering $q \rightarrow 0$, there are two observations. 1) The number of triangles generated by WS is generally more than BA, hence choosing $T(v_{i,ws})$ will get a tighter bound. 2) In addition, most of the vertices in WS have equal number of triangles due to low rewiring probability. Let α be the average number of triangles for any vertex in WS given q . Thus:

$$\begin{aligned}
\text{Cluster Coefficient of } \mathcal{C} &\geq \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ws})}{\binom{\deg(v_{i,c})}{2}} \\
&\approx \frac{\alpha}{n} \sum_{i=0}^n \frac{1}{\binom{\deg(v_{i,c})}{2}} \\
&= \alpha \sum_{k=0}^n P^{\mathcal{C}}(k) \frac{1}{\binom{k}{2}}
\end{aligned}$$

From [15], the cluster coefficient of WS decreases at the rate $(1-q)^3$ and $\alpha \approx (w/2)^2$ for $q = 0$. This is also the rate of decrease to the number of triangles as $(1-q)^3$ is the probability that none of the edges of a triangle is rewired. Thus the lower bound cluster coefficient of the composite network:

$$\text{Cluster Coefficient of } \mathcal{C} \geq \frac{(1-q)^3 w^2}{4} \sum_{k=0}^n P^{\mathcal{C}}(k) \frac{1}{\binom{k}{2}} \quad (18)$$

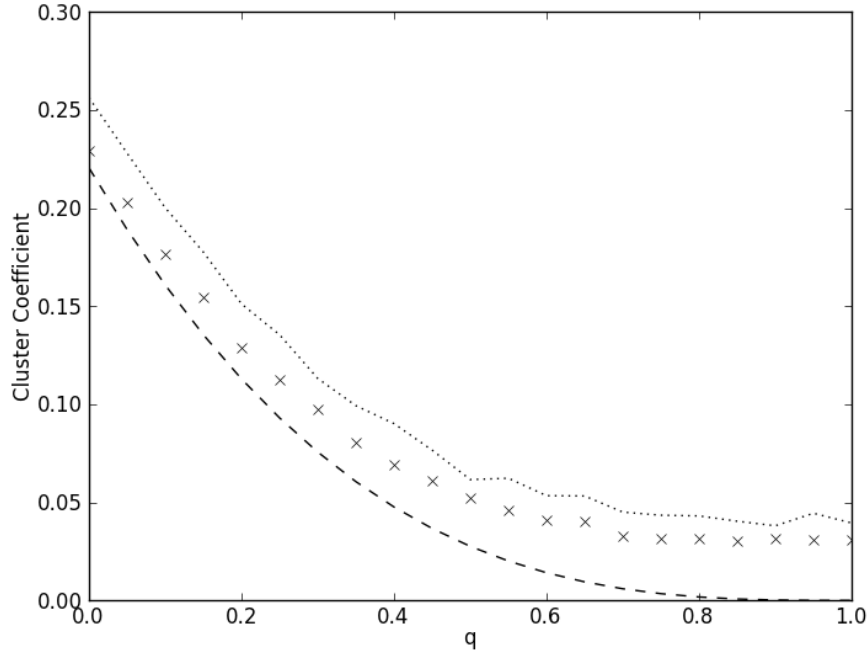


Figure 6: Parameters: $n = 1000$, $w = 10$ and $m = 5$ in blue. The crosses plot the cluster coefficient from the simulation of the composite network. The dashed line is the lower bound from equation 18. The dotted line is the upper bound from equation 19

For the upper bound of the cluster coefficient, we have to round up the cluster coefficient contribution by BA. I.e.

$$\text{Cluster Coefficient of } \mathcal{C} \approx \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ws}) + T(v_{i,ba})}{\binom{\deg(v_{i,c})}{2}}$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ws})}{\binom{\deg(v_{i,c})}{2}} + \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ba})}{\binom{\deg(v_{i,c})}{2}} \\
&\leq \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ws})}{\binom{\deg(v_{i,c})}{2}} + \frac{1}{n} \sum_{i=0}^n \frac{T(v_{i,ba})}{\binom{\deg(v_{i,ba})}{2}} \\
&\approx \frac{(1-q)^3 w^2}{4} \sum_{k=0}^n P^C(k) \frac{1}{\binom{k}{2}} \\
&\quad + \text{Cluster Coefficient of } B_m.
\end{aligned} \tag{19}$$

6. General Observations

Our study is not exhaustive because of the large parameters space. The main constraint on the discussion in the present paper consist in the assumption that the two networks being added together have approximately equal number of vertices and edges. When this assumption is fulfilled we have obtained the following insights.

The first observation is that the union with lattice-like WS, yields a composite network with cluster coefficient higher than the other combinations (Figure 7). This is because there are more triangles at the vertices in a WS network.

For example the cluster coefficient of $v_{i,er}$, $v_{i,ba}$ and $v_{i,ws}$ be proper lowest-term fraction a/b , c/d and e/f . The numerator is the number of triangles and the denominator is the number of triples. Since the cluster coefficient of $ER < BA < WS$, we have

$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$$

Assuming there is no common edge, the cluster coefficient of the composite vertex $v_{i,c}$ is the ratio of the total number of triangles to the total number of triples. i.e. for union of ER and BA, the cluster coefficient is $(a+c)/(b+d)$. By the mediant inequality,

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} < \frac{c+e}{d+f} < \frac{e}{f}$$

Hence unions with lattice-like WS (i.e. $(c+e)/(d+f)$) will yield higher cluster coefficient than other combinations. In fact given equal number of edges, the cluster coefficient of $WS \cup WS$ is greater than $WS \cup BA$, which is greater than $WS \cup ER$ (See Fig. 7).

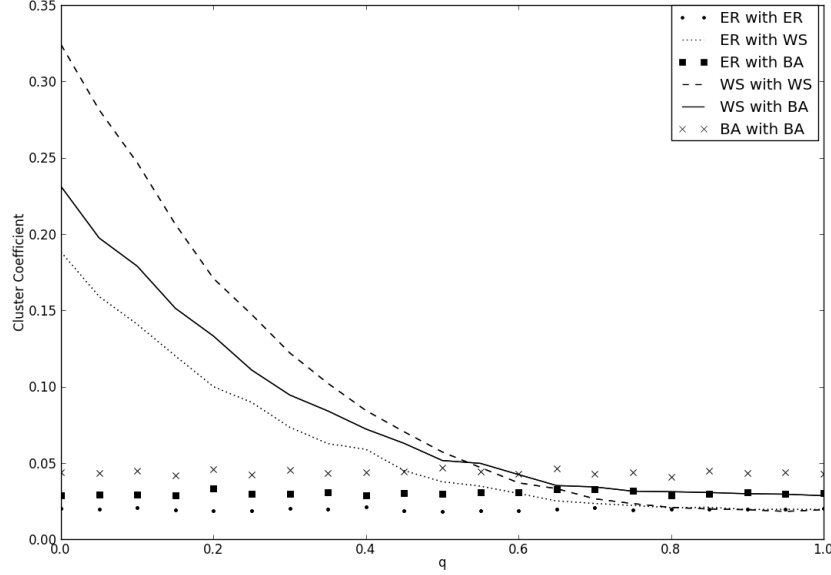


Figure 7: Simulations parameters: $n = 1000$, $w = 10$, $m = 5$ and $p = k/(n - 1)$. The x-axis varies the rewiring probability (q) of WS. For small q , combinations with WS tends to be high. Furthermore, the cluster coefficient of $WS \cup WS$ is greater than $WS \cup BA$, which is greater than $WS \cup ER$

Lastly we observe that unions with BA has power-law-like degree distribution for large degree. The intuition is that high degree vertices in BA has high probability to have common edges with the second network. Hence the percentage change for high degree vertices is low, retaining the same distribution as the original BA.

Secondly, to increase the probability for having a significant number of non-common edges in the second network, the corresponding vertex has to be of high degree too. Since high degree vertices in ER or WS occurs in minute probability, it is unlikely to increase the degree of the vertices in BA. Figure 8 is a degree-rank plot of the different combination of composite networks.

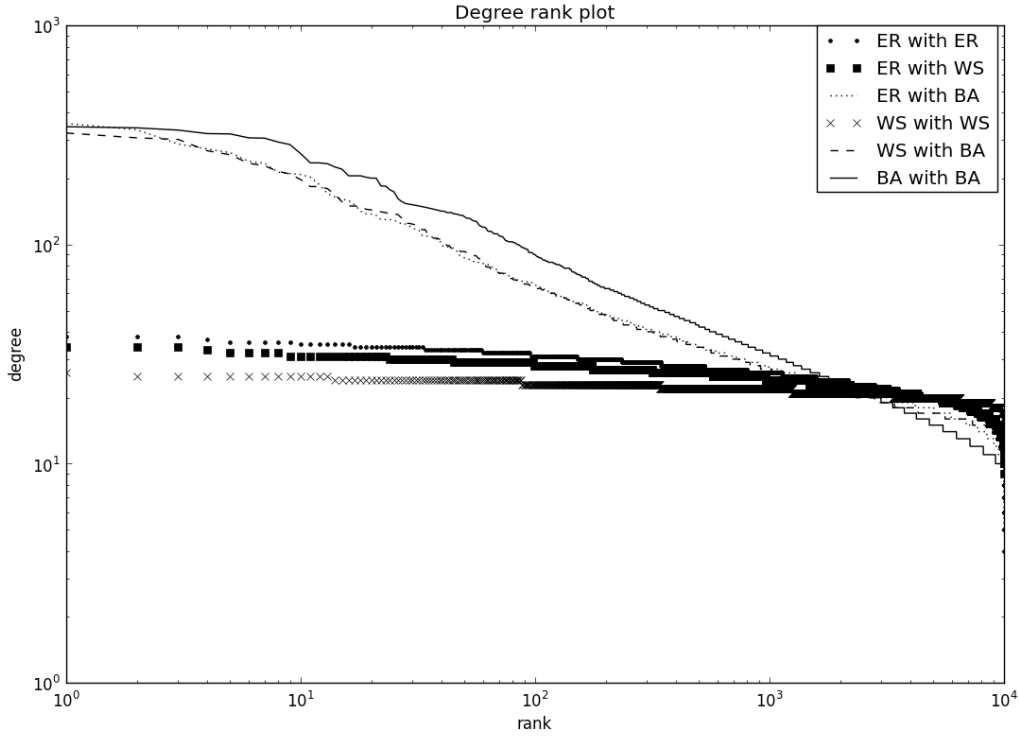


Figure 8: Same parameters as figure 7 with $q = 0.1$. Combinations with BA have power-law-like distribution. Note that $WS \cup BA$ is almost on the same line as $ER \cup BA$. Lastly, $BA \cup BA$ has slight different distribution than the other BA combinations.

Ref. [16] showed that subnets (subgraph) of a scale-free (power-law) network are not scale-free (power-law). Incidentally our result is related. Namely, in a composite power-law-like network \mathcal{C} (i.e. $BA \cup BA$), there exists a power-law subnet, specifically the BA network.

7. Applications

Zachary Karate Club Network[17] is an example of edge union of 8 networks on the same vertex set. Any two members (vertices) of the karate club are connected if they have consistent interactions outside classes and club meetings.

“Outside interactions” are defined by 8 contexts to form the 8 networks on the same vertex set. One of such contexts is the association in and between academic classes at the university [17].

It is common for real-world networks to have similar “layered dimensions”. In temporal networks where connections vary across time, each layer is a time snapshot of the connections of the same vertex set [18]. The edge union of the network snapshots is analogous to extending the time interval in the sampling, i.e. aggregation of a temporal stream of edges over a fixed number of time distance [19].

Aggregation helps to identify the optimal size of the time interval at which to resolve a temporal network. For example assume that vertex v_i and v_j are connected over time T with probability $1 - \exp(-pT)$ where $p \in [0, 1]$ is an arbitrary constant. As we extend (aggregate) the time to $2T$, Eq. (7) gives the edge probability that v_i and v_j are connected over time $2T$ is $1 - \exp(-2pT)$. This result is the same as we would get from Eq. 7 if we add the network at T to the one developed over time T to $2T$.

Lastly the degree distribution of the unions have a lognormal-like distribution. Such distribution can be observed in real networks such as social networks [20] and Protein-Interactions [21]. In fact the Multiplicative Attribute Graph Model in [20] is similar to a multi-dimensional network where each attribute is a dimension. The collection of attribute networks form our networks on the same vertex set.

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Appendix A. Simulation Details

Simulations are implemented in Python 2.7. This is to make use of the existing numerical and network libraries like numpy and NetworkX to minimize implementation errors. The trade off is slower runtime, resulting in smaller sample size.

For simulations, two networks with the same number of vertices are generated with NetworkX network library. The size of the vertex set is chosen such that we can maximize the sample size while computationally feasible (less than 1 hour). For instance the size of the vertex set in Fig. 2 is 10000 as the iterative method (equation 12) is computationally expensive. In contrast, to generate figure 3, it is possible to use vertex set of size 100000.

The size of the edge set also determines the computation costs. In this paper, we chose the minimum size edge set such that the generated networks are connected. The decision for studying connected network is arbitrary from an analysis point of view. However it is useful if we want to study other metrics like average shortest distance in the future.

To ensure ER is a connected network, we need the probability, p , that two vertices are connected to satisfy $p > \ln n/n$. Furthermore in this paper we only consider the case where both networks have approximately equal number of edges. Thus for WS and BA to have approximately equal edge set size as ER, $w \approx (n - 1)p$ and $m \approx (n - 1)p/2$ respectively.

We note that unequal size edge sets introduce additional complexity to the study, since the larger edge set will dominate the resultant network. Ignoring this aspect significantly reduces the complexity of the computation.